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REPORT R-199 APRIL, 1964

COORDINATED SCIENCE LABORATORY
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS

Contract DA-36-039-AMC 02208(E)
DA Project 3A-99-25-004

The research reported in this document was made possible by support extended to the University of Illinois, Coordinated Science Laboratory, jointly by the Department of the Army, Department of the Navy (Office of Naval Research), and the Department of the Air Force (Office of Scientific Research) under Department of Army Contract DA-36-039-AMC 02208(E).

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ABSTRACT

The optimal singular control for a useful class of systems is obtained. Linear, stationary systems with multiple inputs when subject to performance indices quadratic in the state variables, but explicitly independent of the control variables may be optimally governed by singular control. The means of obtaining the singular behavior and control of derived and an explicit formula for the singular control is provided.

I. INTRODUCTION

1. Background

When the performance index is linear in the control variables, or when the control variables do not appear explicitly at all in the performance index, the optimal control (for bounded inputs) is a combination of maximum effort (bang-bang) and singular [1,2]. The means of obtaining the maximum effort controls are well-known, but only recently have the singular solutions received any attention. Johnson and Gibson [1] have obtained an implicit set of equations for the singular solutions when the performance index is explicitly independent of the time variable. Moreover, Wonham and Johnson [2] subsequently solved the special case of a single-input, linear, stationary system subject to a performance index quadratic in the states but independent of the control; they obtained only the stable singular control surface (whereon the system point goes stably to the origin of the state space as $t \to \infty$). In this problem we solve the more general problem by finding all of the singular solutions for multi-input systems subject to quadratic performance indices.

2. Problem Formulation

Consider a linear, stationary, multi-input plant described by the state equations

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \,, \tag{1}$$

where the performance index is

$$J(\underline{u}) = \int_{o}^{T} \underline{x}^{T} Q\underline{x} \, dt. \tag{2}$$

The optimal control problem is to choose a control $\underline{u}(t)$ which takes the system (1) from initial state $\underline{x}_o$ to final state $\underline{x}_T$ while minimizing the functional (2). In particular, we consider only those trajectories which are associated with bounded controls[1], i.e.,

$$\|\underline{u}\| < \infty \tag{3}$$

II. THE OPTIMAL (INTERIOR) CONTROL

1. Two Point Boundary Value Problems

In this section we reduce the above problem to a two point boundary value problem by means of Pontryagin's Maximum Principle [3]. Suppose the rank of the matrix B is m, $0 \leq m \leq n$; then let M be the normalized modal matrix which takes B into Jordan cononical form:

$$M^{-1} BM = D. \tag{4}$$

Since the rank of B is m, so is that of D, and it can be partitioned as

$$D = \begin{bmatrix} \underset{\sim}{0}_{11} & \underset{\sim}{0}_{12} \\ \underset{\sim}{0}_{21} & D_{22} \end{bmatrix}, \tag{5}$$

where

$$\underset{\sim}{0}_{11} \text{ is } (n-m) \times (n-m), \tag{6a}$$

$$\underset{\sim}{0}_{12} \text{ is } (n-m) \times m, \tag{6b}$$

$$\underset{\sim}{0}_{21} \text{ is } m \times (n-m), \tag{6c}$$

and

$$D_{22} \text{ is } m \times m \tag{6d}$$

and nonsingular. Now let

$$\underline{x} = M\underline{y}, \tag{7a}$$

$$M^{-1} AM = C \tag{7b}$$

and

$$\underline{u} = M\underline{v}; \tag{7c}$$

under this transformation of variables (1) becomes

$$\dot{\underline{y}} = C\underline{y} + D\underline{v}. \tag{8}$$

Upon letting

$$\underline{y} = \begin{bmatrix} \underline{z} \\ \\ \underline{w} \end{bmatrix}, \tag{9a}$$

where

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-m} \end{bmatrix} \tag{9b}$$

and

$$\underline{w} = \begin{bmatrix} y_{n-m+1} \\ \vdots \\ y_n \end{bmatrix}, \tag{9c}$$

and

$$\underline{v} = \begin{bmatrix} \underline{\zeta} \\ \\ \underline{\delta} \end{bmatrix}, \tag{10a}$$

where

$$\underline{\zeta} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-m} \end{bmatrix} \tag{10b}$$

and

$$\underline{\delta} = \begin{bmatrix} v_{n-m+1} \\ \vdots \\ v_n \end{bmatrix}, \tag{10c}$$

we can write in place of (8)

$$\dot{\underline{z}} = C_{11}\,\underline{z} + C_{12}\,\underline{w} \tag{11a}$$

and

$$\dot{\underline{w}} = C_{21}\,\underline{z} + C_{22}\,\underline{w} + D_{22}\,\underline{\delta}. \tag{11b}$$

The matrices C_{ij} are obviously obtained from the simple partitioning of C which follows:

$$C = \begin{array}{cl} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & \underbrace{C_{22}}_{\text{m columns}} \end{bmatrix} & \begin{array}{l} \} \text{ n-m rows} \\ \\ \end{array} \end{array} . \tag{12}$$

In order to apply the Maximum Principle, we form the Hamiltonian $\mathcal{H}$ [3]

$$\mathcal{H} = \underline{\Psi}^T A\underline{x} + \underline{\Psi}^T B\underline{u} - \underline{x}^T Q\,\underline{x}\ , \tag{13}$$

where

$$\dot{\underline{\Psi}}_i = -\frac{\partial\mathcal{H}}{\partial x_i}\ . \tag{14}$$

Therefore,

$$\dot{\underline{\Psi}} = -A^T\underline{\Psi} + 2\,Q\underline{x}\ . \tag{15}$$

Making the transformation

$$\underline{\Psi} = M\,\phi \tag{16}$$

and letting

$$\underline{\phi} = \begin{bmatrix} \underline{p} \\ \\ \underline{q} \end{bmatrix}, \tag{17a}$$

where

$$\underline{p} = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_{n-m} \end{bmatrix} \tag{17b}$$

and

$$\underline{q} = \begin{bmatrix} \phi_{n-m+1} \\ \vdots \\ \phi_n \end{bmatrix}, \tag{17c}$$

and

$$R = M^T QM, \tag{18}$$

we can write for (15)

$$\dot{\underline{p}} = - C_{11}^T \underline{p} - C_{21}^T \underline{q} + 2R_{11} \underline{z} + 2R_{12} \underline{w} \tag{19a}$$

$$\dot{\underline{q}} = - C_{12}^T \underline{p} - C_{22}^T \underline{q} + 2R_{21} \underline{z} + 2R_{22} \underline{w} \tag{19b}$$

where the R_{ij} are defined analogously to the C_{ij}. According to the Maximum Principle, for an optimum it is necessary that

$$\frac{\partial \mathcal{H}}{\partial u} = 0, \tag{20}$$

which implies

$$B^T \underline{\Psi} = \underline{0}. \tag{21}$$

The singular solutions are those defined by [1]

$$\frac{d^m}{dt^m} (B^T \underline{\Psi}) = 0, \; m=0,1,2,... \tag{22}$$

and

$$\frac{d^m}{dt^m}[\underline{\Psi}^T A\underline{x} - \underline{x}^T Q\underline{x}] = 0,\ m=0,1,\ldots \tag{23}$$

Now,

$$B^T\underline{\Psi} = D^T\underline{\phi} = \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{0} & D_{22} \end{bmatrix} \begin{bmatrix} \underline{p} \\ \underline{q} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ D_{22}\underline{q} \end{bmatrix} \tag{24}$$

Hence if (22) is true

$$D_{22}\frac{d^m}{dt^m}\underline{q} = \underline{0},\ m=0,1,\ldots \tag{25}$$

Since D_{22} is non-singular, (25) has the unique solution

$$\frac{d^m}{dt^m}\underline{q} = 0,\ m=0,1,\ldots\ . \tag{26}$$

Substituting (26) into (18b) and solving for $\underline{w}$ we obtain (R_{22} is symmetric and non-singular)

$$\underline{w} = \frac{1}{2}R_{22}^{-1}[C_{12}^T\underline{p} - 2R_{21}\underline{z}]\ . \tag{27}$$

Substituting (27) into (18a) and (11a), we obtain

$$\dot{\underline{z}} = (C_{11} - C_{12}R_{22}^{-1}R_{21})\underline{z} + \frac{1}{2}C_{12}R_{22}^{-1}C_{12}^T\underline{p} \tag{28a}$$

and

$$\dot{\underline{p}} = (2R_{11} - 2R_{12}R_{22}^{-1}R_{21})\underline{z} - (C_{11}^T - R_{12}R_{22}^{-1}C_{12}^T)p \tag{28b}$$

Equations (28a) and (28b) define a free linear system of order 2(n-m). Here by free system we mean a system independent of the control $\underline{u}$. It is easy to see that the characteristic equation of this system is (c.f., Appendix)

$$\sum_{i=0}^{n-m} b_i(\lambda)^i(-\lambda)^i = 0; \tag{29}$$

i.e., the eigenvalues occur in quadrantal symmetry. Therefore (n-m) of the solutions of (28) are stable and (n-m) are unstable (assuming no imaginary eigenvalues). Our problem can now be stated as a two-point boundary value problem: Given $\underline{z}(0)$ and $\underline{z}(T)$ (2(n-m) end point conditions), solve equations (28a) and (28b) for $\underline{z}(t)$ and $\underline{p}(t)$. Once z(t) and p(t) are known, $\underline{w}(t)$ is obtained from (27); then $\underline{x}(t)$ is obtained from (7b). The control on the singular trajectories is given by

$$\underline{\delta}(t) = D_{22}^{-1} [\dot{\underline{w}}(t) - C_{21} \underline{z}(t) - C_{22} w(t)], \tag{30a}$$

and

$$\underline{\zeta}(t) = \underline{0} . \tag{30b}$$

The actual control signal $\underline{u}(t)$ can be obtained from (7c). The procedure outlined above will provide time functions for the control components and system variables. As is well known, such representations are of little use in practice; it is desirable to obtain $\underline{u}$ as a function of the system state. The following section shows how such a solution can be obtained.

2. Singular Surfaces and Singular Control

In this section we obtain the singular control in a form suitable for feedback implementation. For the sake of simplicity, we assume that the final state is the origin[2], i.e.,

$$\underline{z}(T) = \underline{0} \tag{31a}$$

and

$$\underline{w}(T) = \underline{0} . \tag{31b}$$

In the notation in Appendix A, we write for (28)

$$\dot{\underline{z}} = \alpha_{11}\, \underline{z} + \alpha_{12}\, \underline{p} \tag{32a}$$

$$\dot{\underline{p}} = \alpha_{21}\, \underline{z} + \alpha_{22}\, \underline{p} \tag{32b}$$

Let the independent initial conditions be $\underline{z}(t_o) = \underline{z}_{t_o}$; then define $\phi(t,t_o)$ as the solution of

$$\dot{\phi}(t,t_o) = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \phi(t,t_o) \tag{33a}$$

$$\phi(t,t_o) = I \; . \tag{33b}$$

Now we consider the following partition of the $(n-m) \times (n-m)$ transition matrix $\phi(t,t_o)$:

$$\phi(t,t_o) = \begin{bmatrix} \phi_{11}(t,t_o) & \phi_{12}(t,t_o) \\ \phi_{21}(t,t_o) & \phi_{22}(t,t_o) \end{bmatrix} \; . \tag{34}$$

Obviously, from (33b)

$$\phi_{11}(t_o,t_o) = \phi_{22}(t_o,t_o) = I \tag{35a}$$

$$\phi_{12}(t_o,t_o) = \phi_{21}(t_o,t_o) = 0 \tag{35b}$$

It can be shown easily [4] that

$$\underline{z}(T) = \phi_{11}(T,t_o)\, \underline{z}(t_o) + \phi_{12}(T,t_o)\, \underline{p}(t_o) = \underline{0}; \tag{36a}$$

therefore,

$$\underline{p}(t_o) = - \phi_{12}^{-1}(T,t_o)\, \phi_{11}(T,t_o)\, \underline{z}(t_o) \; . \tag{36b}$$

Hence,

$$\underline{p}(t) = [\phi_{21}(t,t_o) - \phi_{22}(t,t_o)\ \phi_{12}^{-1}(T,t_o)\ \phi_{11}(T,t_o)]\ \underline{z}(t_o) \quad (37)$$

Upon substituting t_o for t in (37), we obtain

$$\underline{p}(t) = -\ \phi_{12}^{-1}(T,t)\ \phi_{11}(T,t)\ \underline{z}(t). \quad (38)$$

From (27),

$$\dot{\underline{w}} = \frac{1}{2} R_{22}^{-1} [C_{12}^{T}\ \dot{\underline{p}} - 2R_{21}\ \dot{\underline{z}}] ;$$

Substituting (28a) and (28b) into the above expression, we obtain

$$\dot{\underline{w}} = \frac{1}{2} R_{22}^{-1} [C_{12}^{T}\ \alpha_{21}\ \underline{z} + C_{12}^{T}\ \alpha_{22}\ \underline{p} - 2R_{21}\ \alpha_{11}\ \underline{z} - 2R_{21}\ \alpha_{12}\ \underline{p}] \quad (39)$$

Substituting (39) and (27) into (30a), we find

$$\underline{\delta}(t) = D_{22}^{-1} \left[\left\{ \frac{1}{2} R_{22}^{-1} (C_{12}^{T}\ \alpha_{21} - 2R_{21}\ \alpha_{11}) \right\} - C_{21} + C_{22}\ R_{22}^{-1}\ R_{21} \right] \underline{z}$$

$$+ D_{22}^{-1} \left[\left\{ \frac{1}{2} R_{22}^{-1} (C_{12}^{T}\ \alpha_{22} - 2R_{21}\ \alpha_{12}) \right\} - \frac{1}{2} C_{22}\ R_{22}^{-1}\ C_{12}^{T} \right] \underline{p} \quad (40)$$

Equations (38) and (40) show $\underline{u}(t)$ as a linear (possibly time-varying) function of the system states. If the final time is allowed to be infinite, then the ratio $\phi_{12}^{-1}(T,t)\ \phi_{11}(T,t)$ will become time independent and we can write (for (38))

$$\underline{p}(t) = L\ z(t), \quad (41a)$$

where L is an $(n-m) \times (n-m)$ constant matrix defined by

$$L = \lim_{T \to \infty} -\ \phi_{12}^{-1}(T,t)\ \phi_{11}(T,t). \quad (41b)$$

Substituting (38) into (28a), we obtain

$$\dot{\underline{z}} = (C_{11} - C_{12}\ R_{22}^{-1}\ R_{21} - \frac{1}{2} C_{12}\ R_{22}^{-1}\ C_{12}^{T}\ \phi_{12}^{-1}(T,t)\ \phi_{11}(T,t))\ \underline{z}(t) \quad (42)$$

The solutions of the linear (time-invariant if $T \to \infty$), $(n-m)^{th}$ order differential equation (42) are the singular trajectories. The singular surface which passes through the origin (for $T \to \infty$) is given by (27) and (41), i.e.,

$$\underline{w} + (R_{22}^{-1} R_{21} - \frac{1}{2} R_{22}^{-1} C_{12}^{T} L) \underline{z} = 0. \tag{43}$$

(43) is a set of m algebraic equations in the n unknowns $(y_1, y_2, \ldots, y_n)$; each of these quations describes a hyperplane of order n-1 in the n-dimensional y-state space; the intersection of the m (n-1)-dimensional hyperplanes is in general the n-m dimensional singular hyperplane in the state space.

III. EXAMPLES

1. Consider the third order linear system described by

$$\underline{\dot{x}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underline{u} , \tag{44}$$

let the performance index be

$$S(\underline{u}) = \int_{o}^{\infty} (x_1^2 + x_2^2 + x_3^2)\, dt . \tag{45}$$

The final state is the origin; the problem is to find the singular solutions. According to the preceding notation, M = I and, hence,

$$\underline{x} = \underline{y},\ \underline{u} = \underline{v},\ \underline{\Psi} = \underline{\phi} . \tag{46}$$

The rank of B is obviously m = 2; the matrices in (28) are then

$$C_{11} = 0,\ C_{12} = [1\ 0],$$

$$C_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \text{and } C_{22} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} .$$

Furthermore,

$$(Q = R)$$

$$Q_{11} = 1,\ Q_{12} = [0\ 0],$$

$$Q_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ Q_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

Hence, equations (28) yield

$$\dot{z}_1 = p_1$$

and

$$\dot{p}_1 = z_1 ,$$

or

$$\dot{x}_1 = \frac{1}{2}\,\Psi_1 \qquad (47a)$$

and

$$\dot{\Psi}_1 = 2\,x_1 \; . \qquad (47b)$$

Moreover from (27)

$$\underline{w} = \begin{bmatrix} p_1 \\ \\ 0 \end{bmatrix} \; ;$$

i.e.,

$$x_2 = \Psi_1 = 2\,\dot{x}_1 \qquad (48a)$$

and

$$x_3 = 0 \; . \qquad (48b)$$

For $x_1\ (\infty) = 0$, equation (42) yields

$$\dot{x}_1 = -\,x_1 \; ; \qquad (49)$$

(49) together with (48) describes the singular trajectories. The singular surface is given by (43):

$$x_2 + x_1 = 0 \qquad (50a)$$

and

$$x_3 = 0 \; . \qquad (50b)$$

Finally, the singular control is

$$u_1 = 0 \; , \qquad (51a)$$

and

$$u_2 = x_1\ (\, = -\,x_2) \; , \qquad (51b)$$

$$u_3 = 0 \; . \qquad (51c)$$

2. Let the system equation be

$$\dot{\underline{x}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underline{u} \ , \tag{52}$$

while the performance index is still given by (45). Again, M = I; however, m = rank of B = 1. The matrices C_{ij} are

$$C_{11} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \ C_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$C_{21} = (0,0) \text{ and } C_{22} = 0 .$$

From (28),

$$\dot{\underline{z}} = \begin{pmatrix} z_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2}\, p_2 \end{pmatrix} \tag{53a}$$

and

$$\dot{\underline{p}} = \begin{pmatrix} 2z_1 \\ 2z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -p_1 \end{pmatrix} . \tag{53b}$$

The solution of (53) (for $T \to \infty$), $\underline{x}(T) = 0$) is

$$\dot{\underline{z}} = \begin{pmatrix} 0 & -1 \\ -1 & -\sqrt{3} \end{pmatrix} \underline{z} \tag{54}$$

and

$$\underline{p} = \begin{pmatrix} -2\sqrt{3} & -2 \\ -2 & -2\sqrt{3} \end{pmatrix} \underline{z} . \tag{55}$$

From (27),

$$\underline{w} = x_3 = \frac{1}{2}\, p_2 = - z_1 - \sqrt{3}\, z_2 . \tag{56}$$

The singular trajectories are specified by (54) and (56). The singular surface is obtained from (43) as ($z_1 = x_1$, $z_2 = x_2$, $w_1 = x_3$)

$$x_3 - \frac{1}{2}\,(0,1)\begin{pmatrix} -2\sqrt{3} & -2 \\ -2 & -2\sqrt{3} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0\ ,$$

i.e.,

$$x_3 + \sqrt{3}\,x_2 + x_1 = 0. \tag{57}$$

Finally, the singular control is given by (40):

$$\delta_1 = u_3 = \sqrt{3}\,x_1 + 2x_2\ . \tag{58}$$

The control (58) is unstable; for implementation purposes, a stable control is necessary. Such a control can be obtained easily by using the system equation (on the singular surface). The reader may verify that any control

$$u_3 = -\,K\,x_1 - (K\sqrt{3}+1)\,x_2 - (K+\sqrt{3})\,x_3, \tag{59}$$

where $K > 0$, renders the system stable and is identical to (58) on the singular surface.

IV. CONCLUSIONS

The singular control has been obtained for a useful class of optimization problems. Since this most important control can often be employed to obtain a practical optimization of systems even when they are not displaying singular behavior [5], the results of this paper should find wide use.

REFERENCES

1. C. D. Johnson and J. E. Gibson, "Singular Solutions in Problems of Optimal Control," IEEE Transactions on Automatic Control, vol. AC-8, pp. 4-15; January 1963.

2. W. M. Wonham and C. D. Johnson, "Optimal Bang-Bang Control with Quadratic Performance Index," Proc. Fourth Joint Automatic Control Conference, pp. 101-112; 1963.

3. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience Publishers, New York, pp. 17-21; 1962.

4. G. C. Collina and P. Dorato, " Applications of Pontryagin's Maximum Principle: Linear Control Systems," Research Report No. 10-15-62, Polytechnic Institute of Brooklyn, Micro-wave Research Institute; June 1962.

5. R. A. Rohrer and M. Sobral, Jr., "Optimal Linear Switching for Singular Linear Systems," Report R-196, Coordinated Science Laboratory, University of Illinois, Urbana, Illinois; March 1964.

APPENDIX A

In the following the eigenvalues of the 2(n-m)-order linear system defined by (28a) and (28b) are shown to occur in quadrantal symmetry. To prove this let

$$\alpha_{11} = C_{11} - C_{12} R_{22}^{-1} R_{21}, \tag{A.1}$$

$$\alpha_{12} = \frac{1}{2} C_{12} R_{22}^{-1} C_{12}^{T} , \tag{A.2}$$

$$\alpha_{21} = 2R_{11} - 2R_{12} R_{22}^{-1} R_{21} \tag{A.3}$$

$$\alpha_{22} = - (C_{11}^{T} - R_{12} R_{22}^{-1} C_{12}^{T}) . \tag{A.4}$$

First note that

$$\alpha_{12}^{T} = \alpha_{12} , \tag{A.5}$$

$$\alpha_{21}^{T} = \alpha_{21} , \tag{A.6}$$

and

$$\alpha_{22} = - \alpha_{11}^{T} \tag{A.7}$$

(the last equation arises from the symmetry of R - - $R_{21}^{T} = R_{12}$). In the new notation, equations (28) are written as

$$\dot{\underline{z}} = \alpha_{11} \underline{z} + \alpha_{12} \underline{p} \tag{A.8}$$

and

$$\dot{\underline{p}} = \alpha_{12} \underline{z} - \alpha_{11}^{T} \underline{p} . \tag{A.9}$$

Upon defining the matrix

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} , \tag{A.10}$$

we want to show that

$$\det\,[\alpha - \lambda I] = \det\,[\alpha + \lambda I] \tag{A.11}$$

Indeed

$$\det\,[\alpha - \lambda I] = \det\begin{bmatrix} \alpha_{11} - \lambda I_{11} & \alpha_{12} \\ \alpha_{21} & -\alpha_{11}^T - \lambda I_{22} \end{bmatrix}; \tag{A.12}$$

But

$$\det\begin{bmatrix} \alpha_{11} - \lambda I_{11} & \alpha_{12} \\ \alpha_{21} & -\alpha_{11}^T - \lambda I_{22} \end{bmatrix} = \det\begin{bmatrix} -\alpha_{21} & \alpha_{11}^T + \lambda I_{22} \\ \alpha_{11} - \lambda I_{11} & \alpha_{12} \end{bmatrix}$$

$$= \det\begin{bmatrix} \alpha_{11}^T + \lambda I_{22} & \alpha_{21} \\ \alpha_{12} & -\alpha_{11} + \lambda I_{11} \end{bmatrix} \tag{A.13}$$

(upon the interchange of rows and then columns). Taking the transpose of the last matrix in (A.13), we obtain

$$\det\begin{bmatrix} \alpha_{11}^T + \lambda I_{22} & \alpha_{21} \\ \alpha_{12} & -\alpha_{11} + \lambda I_{11} \end{bmatrix} = \det\begin{bmatrix} \alpha_{11} + \lambda I_{22} & \alpha_{12}^T \\ \alpha_{21}^T & -\alpha_{11} + \lambda I_{11} \end{bmatrix}$$

$$= \det\begin{bmatrix} \alpha_{11} + \lambda I_{11} & \alpha_{12} \\ \alpha_{21} & -\alpha_{11} + \lambda I_{22} \end{bmatrix} \tag{A.14}$$

(in view of (A.5) and (A.6)).

But, obviously,

$$\det\begin{bmatrix} \alpha_{11} + \lambda I_{11} & \alpha_{12} \\ \alpha_{21} & -\alpha_{11} + \lambda I_{22} \end{bmatrix} = \det\,[\alpha + \lambda I],\ \text{Q.E.D.} \tag{A.15}$$

FOOTNOTES

1. (3) implies that $\underline{u}$ is bounded, but not necessarily uniformly bounded in t (in other words, it is not necessary that there exists a number M such that for all t, $\|\underline{u}\| \leq M$). The specific constraints on $\underline{u}$ of course determine the nonsingular behavior (e.g., bang-bang control), but within the interior of the control space it is the singular control which we are considering which holds regardless of the constraints.

2. It can always be made so by a simple translation of the state coordinate axes.

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